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Introduction to Computed tomography

Part III: Tomography Reconstruction

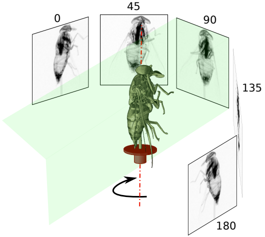
Outline

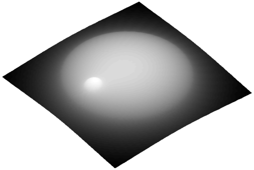
Learning objectives

- Understanding the sinogram
- How projections are related to slices
- Different reconstruction techniques
- Reconstruction filters

Short recap

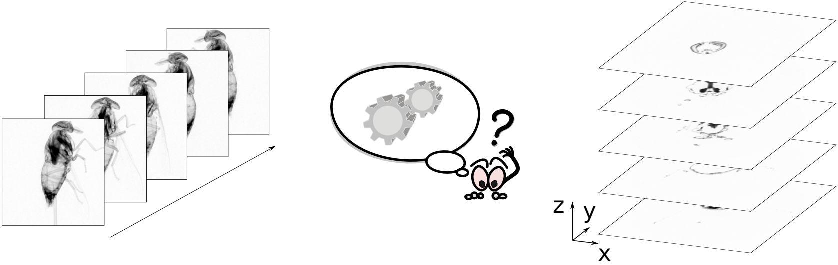
- Acquisiton from different views give depth information
- Reconstruction is not trivial





The reverse process – reconstruction

The scanning provides projection data. . .

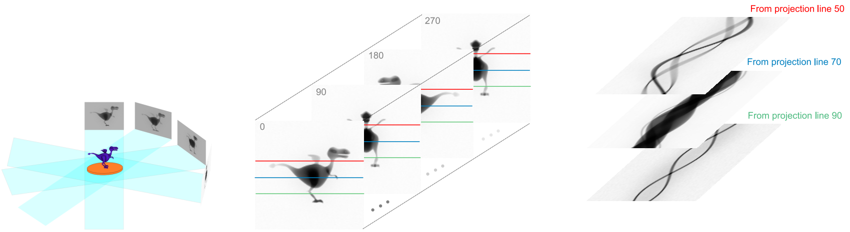


. . . but we want to find the cross section which caused the projection.

> *We have to find the inverse Radon transform or solve the equation system* $A\,x = y$

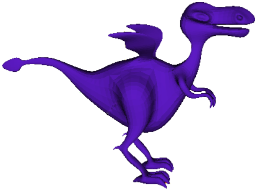
Sinogram construction

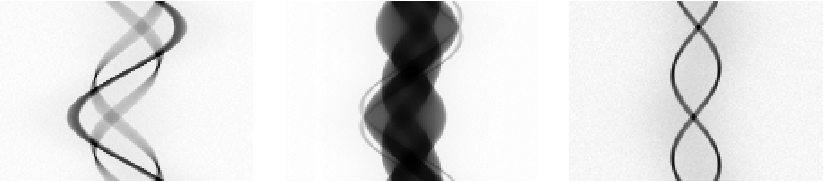
Combine take the same line from all projections into a new image



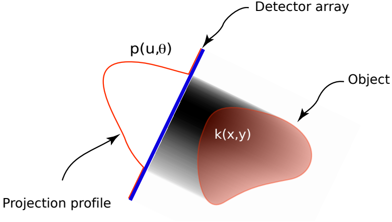
The information required to reconstruct a single slice.

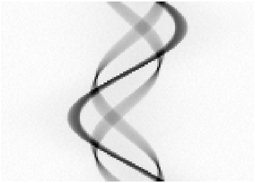
Looking at the sinogram





Projection and sinogram





The Radon transform

An analytical description of projection I acquired at angle θ

$$p = \underbrace{-\ln\left(\frac{I(u,\theta)}{I_0(u)}\right)}_{\text{Measured}} = \int_{-\infty}^{\infty} \underbrace{k(x,y)}_{\text{Wanted}} \underbrace{\delta(x\ \cos\,\theta + y\ \sin\,\theta - u)}_{\text{Observation ray}}\, dx\, dy$$

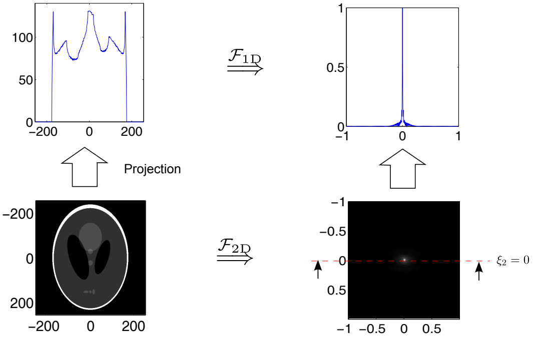
[Radon, 1917]

Inversion – Fourier slice theorem

Theorem

The Fourier transform of a parallel projection $p(x)$ of an object $f(x, y)$ obtained at an angle θ equals a line through origin in the 2D Fourier transform of $f(x, y)$ at the same angle.

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy \qquad f(\xi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi\xi x}\, dx\, dy$$



$$f(x,y) \qquad F(\xi_1,\xi_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi\,(x\xi_1 + y\xi_2)}\, dxdy$$

[Bracewell, 1956]

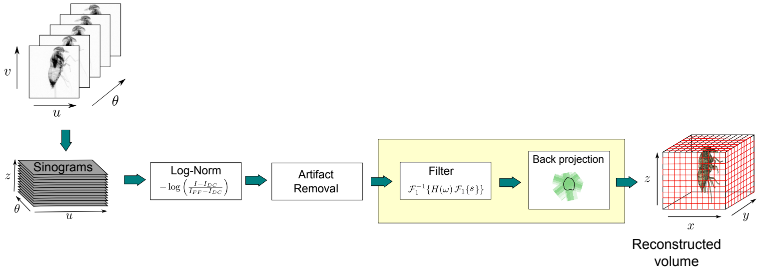
Analytical solution

Reconstruction in the frequency domain

$$k(x,y) = \int_0^{\pi} \int_{-\infty}^{\infty} |\omega| P(\omega,\theta)\, e^{j2\pi\omega(x\cos\theta + y\sin\theta)}\, d\omega d\theta$$

Reconstruction in the spatial domain

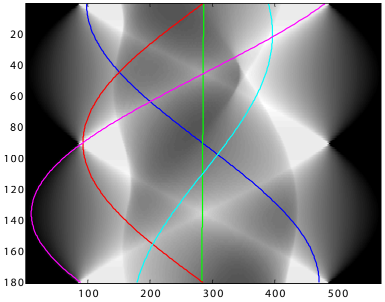
$$k(x,y) = \frac{1}{2\pi^2} \int_0^{\pi} \int_{-\infty}^{\infty} \underbrace{\partial p/\partial u(u,\theta)}_{\textit{Convolution}} \underbrace{[x\cos\theta + y\sin\theta - u]^{-1}}_{\textit{Rotation}}\, du d\theta$$



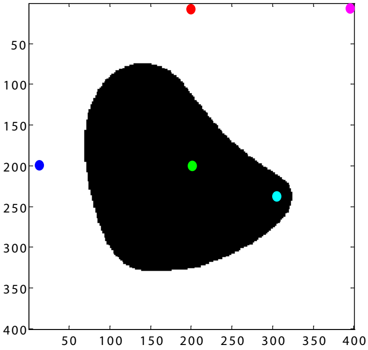
Some line integrals in the sinogram

The value of a single pixel is given by the line integral along a sine.

Sinogram



Cross section



The reconstruction filter

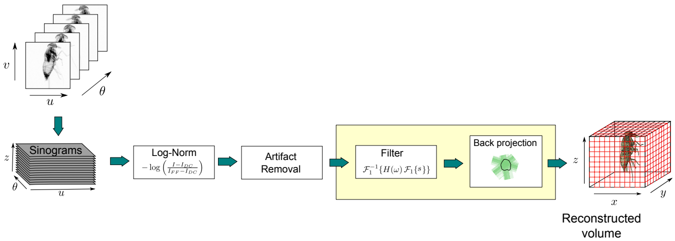
Reconstruction in the spatial domain

$$k(x,y) = \frac{1}{2\pi^2}\int_0^{\pi}\int_{-\infty}^{\infty}\underbrace{\partial p/\partial u(u,\theta)}_{Convolution}\underbrace{[x\cos\theta + y\sin\theta - u]^{-1}}_{Rotation}\,du d\theta$$

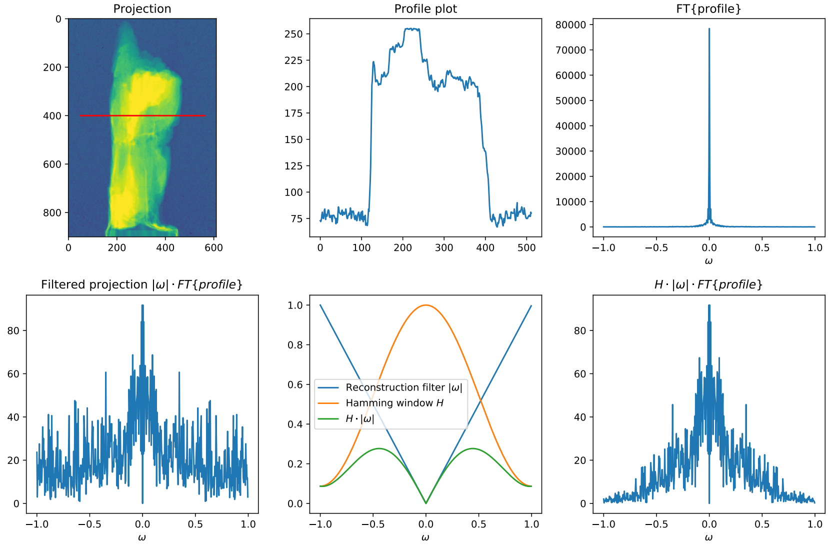
The filter

The filter has two components:

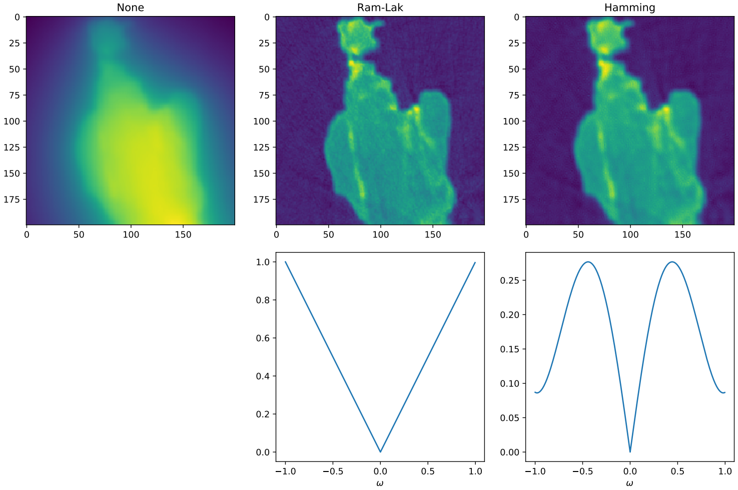
- A derivative: $\partial p/\partial u(u,\theta) \equiv \mathcal{F}^{-1}(|\omega|\cdot\mathcal{F}(p))$
- Apodization: Shepp-Logan, Hamming, *etc*



Reconstruction filter in action



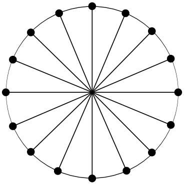
The effect of the reconstruction filter



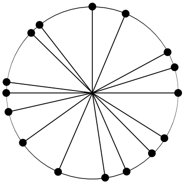
Apodization filters suppress noise and blur edges

When the analytical solution has problems

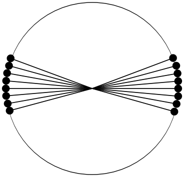
Few projections



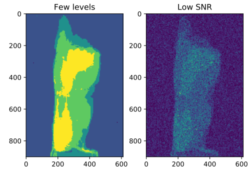
Irregularly distributed



Limited view



Low SNR or contrast



Iterative methods overview

Algebraic methods

- ART
- SIRT
- TV
- etc...

Statistical methods

- Maximum likelihood
- Penalized ML
- etc..

Pros & cons

+ Sparse, irregularly sampled projection data
  - Limited angle
  - Few views
+ Physical model can be included
- Requires prior information for best performance.
- Time consuming

Why iterative inversion

Building the system matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots &= y_1 \\ a_{21}x_3 + a_{22}x_4 + \dots &= y_2 \\ a_{11}x_1 + a_{21}x_3 + \dots &= y_3 \\ &\vdots \end{aligned}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

Example

You have:

- 1000 projections which are 1000 pixels wide
- The reconstructed slice has 1000×1000

This gives $1000 \times 1000 \times 1000 = 10^9$ equations
$\rightarrow$ A is a $10^9 \times 10^9$ matrix!

Some features of A

- Sparse matrix
- Ill-posed (ideally infinitely many equations needed)
- Inversion doesn't provide unique solution

Algebraic Reconstruction Method (ART)

Problem to solve

We want to solve the equation $A\,x = y$,
where A is the forward projection operator, a large sparse matrix

Kaczmarz method (ART)

$$x^{k+1} = x^k + \lambda_k \frac{y_i - \langle a_i, x^k \rangle}{\|a_i\|^2}\, a_i$$

- a_i the i^{th} row of the system matrix A.
- x^k the reconstructed image at the k^{th} iteration.
- y_i the i^{th} element of the sinogram
- λ_k relaxation parameter

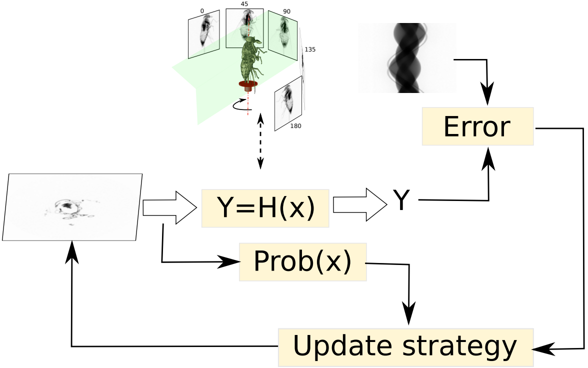
Statistic reconstruction methods

Problem to solve

We want to solve the equation $A\,x = y + noise$,

Iteration scheme

Maximize likelihood function:



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Summary

Reconstrution is

- The process to convert projections into volumes
- Different techniques can be used:
  - Analytical - filtered back projection
  - Algebraic - schemes to solve huge equation systems
  - Statistical - using noise models

References I

Bracewell, R. (1956).
Strip integration in radio astronomy.
Australian Journal of Physics, 9(2):198.

Radon, J. (1917).
Ueber die bestimmung von funktionen durch ihre Integralwerte tangs gewisser mannigfaltigkeiten.
Berichte Saechsisch Akad. Wissenschaften (Leipzig), Math. Phys., Klass 69:262–277.